

This coursework is optional. If you hand in, and get more marks on this than on the worst of your counted required courseworks, the mark for this will be used instead. This only applies if you have completed (or been excused from) at least 7 of the required courseworks. Even if you do not hand in, these examples and the ones in coursework 9 should be useful practice for the examination.

The deadline is 5PM on Tuesday December 12th. Hand-in is to the green box in the basement of the Maths building. Please note that owing to the Xmas holiday of the marker, marked scripts will not be available until next term.

1. For each of the following fields $\mathbf{F}$, evaluate $\nabla \times \mathbf{F}$ and either find the general solution Φ satisfying $\mathbf{F} = \nabla\Phi$ everywhere, or show that no such Φ exists:
 - (a) $\mathbf{F} = z\mathbf{j} + y\mathbf{k}$
 - (b) $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j}$
 - (c) $\mathbf{F} = (y+z)\mathbf{i} + (x-z)\mathbf{j} + (x-y)\mathbf{k}$
 - (d) $\mathbf{F} = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$.

[32]

Answer

In each case, the curl can be calculated directly from the standard formulae, or if you spot that the $\mathbf{F}$ is conservative, you can find Φ and then just say $\nabla \times \mathbf{F} = \nabla \times \nabla\Phi = \mathbf{0}$.

For the conservative fields, Φ can be found by successive integration of components or from the line-integral definition (too many variants to write out!)

- (a) $\Phi = yz + \text{constant}$ and $\nabla \times \mathbf{F} = \mathbf{0}$.
- (b) $\nabla \times \mathbf{F} = -y\mathbf{i} - x\mathbf{k}$. No Φ is possible.
- (c) $\Phi = xy - yz + zx + \text{constant}$ and $\nabla \times \mathbf{F} = \mathbf{0}$.
- (d) $\nabla \times \mathbf{F} = 2\mathbf{k}$. No Φ is possible.

Marks: in each part, 4 for the curl and 4 for Φ .

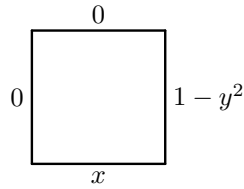
Mark scheme allows for reward for students who compute curls directly when it is not necessary, and conversely with Φ parts.

2. A function $F(x, y)$ satisfies Laplace's equation in the square $x, y \in [0, 1]$ with boundary values as indicated below.

This $F(x, y)$ can be split into two functions

$$F(x, y) = f(x, y) + g(x, y)$$

such that $f(x, y) = 0$ at the corners and $g(x, y) = F(x, y)$ at the corners.



(a) Find $g(x, y)$.

(b) Find $f(x, y)$ on the sides.

[The full solution for $f(x, y)$ is not required for this problem.]

[10]

Answer

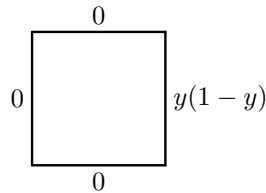
We use the standard trick for eliminating corners: set

$$g(x, y) = \alpha + \beta x + \gamma y + \delta xy$$

and solve for the corner values, which gives

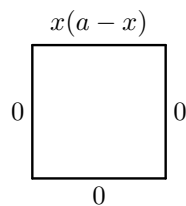
$$g(x, y) = x - xy.$$

The boundary values of $f(x, y)$ are simply the boundary values of $F(x, y) - g(x, y)$, or as shown below



Marks: Correct $g(x, y)$ (5), new boundary conditions (5).

3. Solve Laplace's equation for $f(x, y)$ in the square $x, y \in [0, a]$ with boundary values as indicated below.



[Hint: You may find the result of Example 1.2 helpful]

How does this relate to the $f(x, y)$ still to be found in the previous problem?

[25]

Answer

A solution of the form

$$f(x, y) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/a) \sinh(n\pi y/a)$$

will work. (It satisfies Laplace's equation, and is zero for $x = 0$, $x = a$ and $y = 0$ as required.)

In example 1.2 we showed that on $0 \leq x \leq \pi$,

$$x(\pi - x) = \frac{8}{\pi} \sum_{p=0}^{\infty} \frac{\sin(2p+1)x}{(2p+1)^3}.$$

So changing the length of the range from π to a by scaling x , we have

$$x(a - x) = \frac{8a^2}{\pi^3} \sum_{p=0}^{\infty} \frac{\sin[(2p+1)\pi x/a]}{(2p+1)^3}.$$

Thus b_n is such that

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x/a) \sinh(n\pi) = \frac{8a^2}{\pi^3} \sum_{p=0}^{\infty} \frac{\sin[(2p+1)\pi x/a]}{(2p+1)^3}.$$

Hence the solution is

$$f(x, y) = \sum_{p=1}^{\infty} \frac{8a^2}{(2p+1)^3 \pi^3} \frac{\sinh((2p+1)\pi y/a)}{\sinh((2p+1)\pi)} \sin((2p+1)\pi x/a).$$

This gives the answer to the previous problem if we set $a = 1$ and exchange x and y .

Class Since the necessary Fourier series is given in the notes, encourage them to use the notes to check their answer. and suggest they look at Section 1.5 to see how to adapt this to the problem.

Marks: Correct method (5), right trial solution 5, working out Fourier coefficients correctly (5), correct full solution (5), relation to previous question 5.

4. A potential Φ satisfies Laplace's equation in the region outside a circular cylinder of radius a . Far from the cylinder $\nabla\Phi$ is a constant vector perpendicular to the cylinder's axis. Taking cylindrical polar coordinates (ρ, ϕ, z) , with the z -axis along the axis of the cylinder and $\phi = 0$ in the direction of $\nabla\Phi$ far from the cylinder, Φ is everywhere independent of the coordinate z and satisfies

$$\begin{cases} \partial\Phi/\partial\rho = 0 & \text{on } \rho = a \\ \nabla\Phi \rightarrow \mathbf{u}_{\infty} & \text{as } \rho \rightarrow \infty \end{cases}$$

where $\mathbf{u}_{\infty} = U \cos \phi \mathbf{e}_{\rho} - U \sin \phi \mathbf{e}_{\phi}$. Show that

$$\Phi = U \left(\rho + \frac{a^2}{\rho} \right) \cos \phi + \frac{\kappa}{2\pi} \phi + C$$

is a solution of this problem, for any constants κ and C . Although this solution is multi-valued for $\kappa \neq 0$, show that $\mathbf{u} \equiv \nabla\Phi$ is single-valued everywhere. Show that the integral around a circle of radius $\rho > a$ of $\int \mathbf{u} \cdot d\mathbf{r}$ is equal to κ .

[A physical context in which this arises is when the cylinder is a body at rest in an irrotational, incompressible flow which far away is uniform with magnitude U perpendicular to the axis of the cylinder. κ is the *circulation* of the flow around the cylinder. One can show that the cylinder experiences a force from the fluid proportional to $U\kappa$. Essentially the same result applies to more complex cross-sections e.g. aircraft wings, and if the cylinder is moving and the fluid is at rest; it shows that if an aircraft in flight is stalling (i.e. has inadequate lift), the (counter-intuitive) cure is to dive towards the ground, to increase U .]

Answer

Since we are given the answer, we only need to check it, since the uniqueness theorem guarantees it will then be correct.

We can quote the formula (section 8.4) for $\nabla^2\Phi$ in cylindrical polars, specializing to the z -independent case:

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{\rho} \left[\frac{\partial}{\partial\rho} \left(\rho \frac{\partial\Phi}{\partial\rho} \right) + \frac{\partial}{\partial\phi} \left(\frac{1}{\rho} \frac{\partial\Phi}{\partial\phi} \right) \right] \\ &= \frac{1}{\rho} \left[\frac{\partial}{\partial\rho} \left(U \left(\rho - \frac{a^2}{\rho} \right) \cos\phi \right) + \frac{\partial}{\partial\phi} \left(-U \left(1 + \frac{a^2}{\rho^2} \right) \sin\phi + \frac{\kappa}{2\pi\rho} \right) \right] \\ &= \frac{U}{\rho} \left[\left(1 + \frac{a^2}{\rho^2} \right) \cos\phi - \left(1 + \frac{a^2}{\rho^2} \right) \cos\phi \right] = 0.\end{aligned}$$

So it satisfies the equation. Now we must check the boundary conditions. We implicitly calculated the components of $\nabla\Phi$ in the preceding expressions.

$$\begin{aligned}\nabla\Phi &= U \left(\rho - \frac{a^2}{\rho} \right) \cos\phi \mathbf{e}_\rho - U \left(1 + \frac{a^2}{\rho^2} \right) \sin\phi \mathbf{e}_\phi \\ &\rightarrow U \cos\phi \mathbf{e}_\rho - U \sin\phi \mathbf{e}_\phi \quad \text{as } \rho \rightarrow \infty\end{aligned}$$

In the preceding expression, the ρ component of $\nabla\Phi$ gives us $\partial\Phi/\partial\rho$ and this is 0 at $\rho = a$ as required. So we have now shown the given form satisfies all three conditions and hence is the solution.

Inspection of $\nabla\Phi$ shows it is single valued (i.e. the same at $\phi = 0$ and $\phi = 2\pi$, and continuous in between), unlike Φ itself.

The circulation round the circle, on which $d\mathbf{r} = \rho d\phi \mathbf{e}_\phi$, is

$$\int_0^{2\pi} \nabla\Phi \cdot d\mathbf{r} = \Phi(\rho, 2\pi) - \Phi(\rho, 0) = \kappa.$$

Marks: Correct method 5, check of ∇^2 5, check of asymptotic value 5, check of derivative on $\rho = a$ 3, ∇ single-valued 3, circulation 4.

5. Consider the region $\mathcal{D}$ defined in spherical polar coordinates by $1 \leq r \leq 2$. Find the solution Φ of Laplace's equation, independent of ϕ , which satisfies the following boundary conditions: (i) $\Phi = 3 \cos^2 \theta + \cos \theta - 1$ when $r = 1$ and (ii) $\Phi = 3 \cos \theta$ when $r = 2$. [13]

Answer

Following the idea that we only need in the solution those harmonics which appear in the boundary conditions, and noting that $P_2(\cos \theta) = \frac{1}{2}(3 \cos^2 \theta - 1)$ we guess

$$\Phi = \left(Ar + \frac{B}{r^2} \right) \cos \theta + \left(Cr^2 + \frac{D}{r^3} \right) (3 \cos^2 \theta - 1)$$

(see section 8.5 in the notes for the reasons).

To fit the boundaries we need $A + B = 1$, $2A + B/4 = 3$, $C + D = 1$, $4C + D/8 = 0$, and thus we get $A = 11/7$, $B = -4/7$, $C = -1/31$, $D = 32/31$. Finally we get

$$\Phi = \left(11r - \frac{4}{r^2} \right) \frac{\cos \theta}{7} + \left(\frac{32}{r^3} - r^2 \right) \frac{(3 \cos^2 \theta - 1)}{31}$$

Marks Form of guess 5, getting equations for boundaries 4, solving correctly 4.